CSUSM
STEM Success Center

Math 162 Chapter 11/Sections 1-8 Topic: Series and Sequences Worksheet

1. **Sequence.**
   a. Define what a sequence is.

   b. Define Increasing and decreasing sequences/ Monotonic sequences

   c. State the Monotonic Sequence Theorem

   d. List the fibonacci sequences

   e. Give an example of a Recurrence Relations

   f. Practice Problems:

      a. List first 5 terms for the following sequences

      1. $a_n = \frac{2^n}{2n+1}$

      2. $a_n = \frac{1}{n!}$

      3. $a_n = 5n$

      4. $a_1 = 1, \quad a_{n+1} = 3a_n + 2$

      5. $a_n = \frac{(-1)^n}{n}$

      b. Find a general formula for the following sequences.

      1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\ldots$

2. 1,3,5,7,9,……..

3. 3,6,9,12,15, ……….

4. 1,0,-1,0,1,0,...............

5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$

c. Determine whether if the following sequence converges or diverges, if converges then find the limit.

1. $a_n = \frac{3 + 5n^2}{n + n^2}$

2. $a_n = \frac{3n^2}{n+7}$

3. $a_n = 4 + (0.76)^n$

2. Series.

a. Define Infinite Series

b. Partial Sums

c. Give an example of an Arithmetic Series

d. Give an example of a Geometric Series

e. Give an example of a Telescoping Series

f. Give an example of a Harmonic Series

g. Test for Divergence (Theorem)

h. Practice Problems:

a. Find the first 5 terms of the following series

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{1}{n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

b. Check whether this series converges/diverges. If the series converges then find the partial sum.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{1}{n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

3. Integral Tests

a. Define the Integral Test.

b. Practice Problems:

Use the integral test to determine whether this series converges or diverges

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

3. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$

4. Comparison tests

a. Define what a comparison test is.

b. Define the Limit Comparison test

c. Practice Problems:

Use the comparison test or the Limit comparison test to determine whether the series converges/diverges. If the series converges then find the partial sum.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$

3. $\sum_{n=1}^{\infty} \frac{1}{12n+5}$

5. Alternating Series

a. Define what alternating series is.

b. Practice Problems:

Determine whether the series converges or diverges.

1. $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots\ldots$

2. $\sum_{n=1}^{\infty} \quad (-1)^{n+1} (\frac{1}{n^2+4})$

3. $\frac{1}{ln(3)} - \frac{1}{ln(4)} + \frac{1}{ln(5)} - \frac{1}{ln(6)} + \ldots.$

6. Absolute Convergence and the Ratio and Roots Tests

a. Define the following terms:

1. Absolute Convergence

2. Ratio Test

3. Root Test

b. Practice Problems:

Determine whether this series converges by using absolute, ratio or root test.

1. $\sum_{n=1}^{\infty} \quad (-1)^{n} (\frac{1}{7n+2})$

2. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

3. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

7. Power Series

a. Define the Power Series.

b. Practice Problems:

Find the radius of convergence or interval of convergence.

1. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

Solutions

1. **Sequence.**

a. Define what a sequence is.

A sequence is a list of number which can be written in a definite order,

$a_1, a_2, a_3, a_4,$ ………………………., a_n, ………………………..

a_1 is called the first term, a_2 is called the second term and a_n is called the last term.

b. Define Increasing and decreasing sequences/ Monotonic sequences

A sequence $\{a_n\}$ is called an increasing sequence when $a_n \leq a_{n+1}$ when $n \geq 1$. For example $a_1 < a_2 < \ldots < a_n$. A sequence is decreasing when $a_n \geq a_{n+1}$. When a sequence is either increasing or decreasing it is monotonic.

c. State the Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

d. List the fibonacci sequences

1,1,2,3,5,8,13,21,……

e. Give an example of a Recurrence Relations

You can have multiple answers. Examples: Fibonacci Sequence, Tower of Hanoi, etc.

f. Practice Problems:

a. List first 5 terms for the following sequences

1. $a_n = \frac{2^n}{2n+1}$

$$\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$$

2. $a_n = \frac{1}{n!}$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}$$

3. $a_n = 5n$

5, 10, 15, 20, 25

4. $a_1 = 1$, $a_{n+1} = 3a_n + 2$

1, 5, 17, 53, 161

5. $a_n = \frac{(-1)^n}{n}$

$$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}$$

b. Find a general formula for the following sequences.

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\ldots$

$$a_n = \frac{1}{2n}$$

2. 1,3,5,7,9,……..

$$a_n = 2n + 1$$

3. 3,6,9,12,15, ……….

$$a_n = 3n$$

4. 1,0,-1,0,1,0,..............

$$a_n = cos(\frac{n\pi}{2})$$

5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$

$a_n = \frac{1}{2^n}$

c. Determine whether if the following sequence converges or diverges, if converges then find the limit.

1. $a_n = \frac{3 + 5n^2}{n + n^2}$

$\lim_{n\to\infty} \frac{10n}{1+2n} = \lim_{n\to\infty} \frac{10}{2}$ by L'Hospital Rule. $\lim_{n\to\infty} \frac{10}{2} = 5$. Hence this sequence converges.

2. $a_n = \frac{3n^2}{n+7}$

$\lim_{n\to\infty} \frac{3n^2}{n+7} = \lim_{n\to\infty} 6n = \infty$. Hence this sequence diverges by L'Hospital rule.

3. $a_n = 4 + (0.76)^n$

$\lim_{n\to\infty} 4 + (0.76)^n = \lim_{n\to\infty} 4 + 0 = 4$. Thus the sequence converges.

2. Series.

a. Define Infinite Series

A sum of infinite sequences $\{a_n\}\infty_{n=1}$ which is expressed in a form as

$a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots$ This is called an infinite series. Or simply it can be denoted in a form $\sum_{\square=1}^{\infty} a_n$.

b. Partial Sums

The sum up to the n^{th} term

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

.

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$s_n = a_1 + a_2 + \ldots\ldots. + a_n$

c. Give an example of an Arithmetic Series

$\sum_{\square=1}^{\infty} 2k = 2 + 4 + 6 + 8 + \ldots\ldots + 2k + \ldots\ldots\ldots\ldots$

d. Give an example of a Geometric Series

$\sum_{\square=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots.. + ar^{n-1} + \ldots$ where as $a \neq 0$ and

r is a common ratio.

$s_n = \frac{a(1-r^{\ \ n})}{1-r}$ only if $-1 < r < 1$.

e. Give an example of a Telescoping Series

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \Sigma\infty_{n=1} \frac{1}{n} - \frac{1}{n+1} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots\ldots.. + (\frac{1}{n} + \frac{1}{n+1}) = \lim_{n\to\infty} 1 + \frac{1}{n+1} = 1$ $- 0 = 1$.

f. Give an example of a Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

g. Test for Divergence (Theorem)

If a limit does not exist or approaches infinity or $\lim_{n\to\infty} a_n \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

h. Practice Problems:

a. Find the first 5 terms of the following series

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}$$

2. $\sum_{n=1}^{\infty} \frac{1}{n}$

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}$$

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$

b. Check whether this series converges/diverges. If the series converges then find the partial sum.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\frac{\pi^2}{6}$. Use a calculator and sum up all the terms. Once you reach a larger number you will get closer and closer to $\frac{\pi^2}{6}$.

2. $\sum_{n=1}^{\infty} \frac{1}{n}$

Since we have an harmonic series this series diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \Sigma\infty_{n=1} \frac{1}{n} - \frac{1}{n+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots.. + \left(\frac{1}{n} + \frac{1}{n+1}\right) =$
$\lim_{n\to\infty} 1 + \frac{1}{n+1} = 1 - 0 = 1$. Since we reach a finite limit, the series converges.

3. Integral Tests

a. Define the Integral Test.

f must be continuous, positive, decreasing function on [1, ∞).

If $\int_1^{\infty} f(x)\,dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent. If $\int_1^{\infty} f(x)\,dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

b. Practice Problems:

Use the integral test to determine whether this series converges or diverges

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+3} = \int_0^{\infty} \frac{1}{x^2+3}dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+3}dx.\ x = u\sqrt{3}.\ dx = \sqrt{3}du.$

$\int_0^t \frac{\sqrt{3}}{(u\sqrt{3})^2+3}du = \int_0^t \frac{\sqrt{3}}{3(u^2+1)}du = \frac{1}{\sqrt{3}}arctan(u) = \frac{1}{\sqrt{3}}arctan(\frac{x}{\sqrt{3}}) = \frac{\pi}{2\sqrt{3}}$. So by integral test, this series converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3} = \int_0^{\infty} \frac{1}{x^2+x^3}dx = \int_0^{\infty} \frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x}\,dx = \frac{-1}{x} + ln(x+1) - ln(x) = \infty.$

Therefore this series diverges by integral test.

3. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4} = \int_0^{\infty} \frac{x^2}{x^3+4}dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^3+4}dx.\ u = x^3 + 4.\ du = 3x^2dx.\ \frac{1}{3}du = x^2dx.$

$\frac{1}{3}ln(u) = \frac{1}{3}ln(x^3+4) = \infty$. Hence the series diverges by integral test.

4. Comparison tests

a. Define what a comparison test is.

For all n ∈N, when $a_n \le b_n$, if $\sum b_n$ is convergent then $\sum a_n$ is also convergent.

For all n ∈N, when $a_n \geq b_n$, if $\sum b_n$ is divergent then $\sum a_n$ is also divergent.

b. Define the Limit Comparison test

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ = c whereas c > 0 and c is a finite number then $\sum a_n$ and $\sum b_n$ converges or $\sum a_n$ and $\sum b_n$ diverges.

c. Practice Problems:

Use the comparison test or the Limit comparison test to determine whether the series converges/diverges. If the series converges then find the partial sum.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

   Since $\frac{1}{n^2+4} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by comparison test this series converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$

   Since $\frac{1}{n^3+5} < \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then by comparison test this series converges.

3. $\sum_{n=1}^{\infty} \frac{1}{12n+5}$

   Observe that $\frac{1}{12n+5} > \frac{1}{12n+12}$ because 12n + 12 > 12n + 5. So $\sum_{n=1}^{\infty} \frac{1}{12n+12} = \frac{1}{12}\sum_{n=1}^{\infty} \frac{1}{n+1}$. So $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$. Since the $\sum_{n=2}^{\infty} \frac{1}{n}$ is a harmonic series, it diverges. So by the comparison test, this series diverges.

5. Alternating Series

a. Define what alternating series is.

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots\ldots\ldots\ldots$$

This only satisfies when for all n ∈ N,

$a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$.

b. Practice Problems:

Determine whether the series converges or diverges.

1. $\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+.......$

$\sum_{n=1}^{\infty} \quad (-1)^{n-1}\frac{1}{2n+1}$. $\lim_{n\to\infty}\frac{1}{2n+1}=\frac{1}{\infty}=0$.Therefore by Alternating Series Test, this series converges.

2. $\sum_{n=1}^{\infty} \quad (-1)^{n+1}(\frac{1}{n^2+4})$

$\lim_{n\to\infty}\frac{n}{n^2+4} = \lim_{n\to\infty}\frac{1}{2n}$by L'Hospital Rule. $\lim_{n\to\infty}\frac{1}{2n}=0$.Therefore this series converges by Alternating Series Test.

3. $\frac{1}{ln(3)}-\frac{1}{ln(4)}+\frac{1}{ln(5)}-\frac{1}{ln(6)}+\ldots.$

$\sum_{n=1}^{\infty} \quad (-1)^{n-1}\frac{1}{ln(n+2)}$. $\lim_{n\to\infty}\frac{1}{ln(n+2)} = 0$. Therefore this series converges by Alternating Series Test.

6. Absolute Convergence and the Ratio and Roots Tests

a. Define the following terms:

1. Absolute Convergence

If $\sum \quad |a \quad _n|$is convergent then $\sum \quad a \quad _n$is convergent.

2. Ratio Test

If $\lim_{n\to\infty}|\frac{a \quad _{n+1}}{a \quad _n}|$= L < 1, then the series $\sum \quad a \quad _n$is absolutely convergent.

If $\lim_{n\to\infty}|\frac{a \quad _{n+1}}{a \quad _n}|$= L > 1, then the series $\sum \quad a \quad _n$diverges.

If $\lim_{n\to\infty}|\frac{a \quad _{n+1}}{a \quad _n}|$= 1, then the Ratio Test is Inconclusive.

3. Root Test

If $\lim_{n\to\infty}\sqrt[n]{a \quad _n}$ = 1, then the Root Test is inconclusive.

If $\lim_{n\to\infty}\sqrt[n]{a \quad _n}$ = L < 1, then the series $\sum \quad a \quad _n$is absolutely convergent.

If $\lim_{n\to\infty}\sqrt[n]{a \quad _n}$ = L > 1 or $\lim_{n\to\infty}\sqrt[n]{|a \quad _n|} = \infty$, then the series $\sum \quad a \quad _n$is divergent.

b. Practice Problems:

Determine whether this series converges by using absolute, ratio or root test.

1. $\sum_{n=1}^{\infty} (-1)^n(\frac{1}{7n+2})$

$\sum_{n=1}^{\infty} |\frac{(-1)^n}{7n+2}| = \sum_{n=1}^{\infty} \frac{1}{7n+2}$. So $\frac{1}{7n+2} > \frac{1}{7n+7} . \frac{1}{7}(\frac{1}{n+1})$. So since $\frac{1}{n}$is a harmonic series, the series diverges.

2. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

$\lim_{n\to\infty} |\frac{\frac{(n+1)}{5^{(n+1)}}}{\frac{n}{5^n}}| = \lim_{n\to\infty} |\frac{n+1}{5^{n+1}} * \frac{5^n}{n}| = \frac{5^n}{5^n*5} \lim_{n\to\infty} |\frac{n+1}{n}| = \frac{1}{5} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{5} * 1 = \frac{1}{5} < 1.$

3. $\sum_{n=1}^{\infty} (\frac{-2n}{n+1})^{5n}$

$\lim_{n\to\infty} \sqrt[5n]{|a_n|} = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} 2 = 2$. Since 2 > 1 this series diverges by root test.

7. Power Series

a. Define the Power Series.

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots$

b. Practice Problems:

Find the radius of convergence or interval of convergence.

1. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \lim_{n\to\infty} |\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}| = \lim_{n\to\infty} |\frac{x^{2n+2}}{(n+1)(n)!} * \frac{n!}{x^{2n}}| = x^2 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Therefore the Radius of convergence is $-\infty < x < \infty$.